Advanced ODE-Lecture 1
Some Often Used Fundamental Math Tools
Banach Fixed Point Theorems

Dr. Zhiming Wang

Professor of Mathematics
East China Normal University, Shanghai, China

Advanced ODE Course
September 23, 2014

Outline

- Motivation
- Banach Space
- Banach Fixed Point Theorem
- Schauder Fixed Point Theorem without Proof
- Summary

Motivation

In an experiment on a physical system we expect:

- **(Existence)** that starting from an initial state $x(t_0)$, the state will move and $x(t)$ will be defined in (at least immediate) future time $t > t_0$;
- **(Uniqueness)** to have exactly the same behavior if we repeat the experiment in the same way.

The mathematical model of such a physical system:

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases} \qquad \text{(IVP)}$$

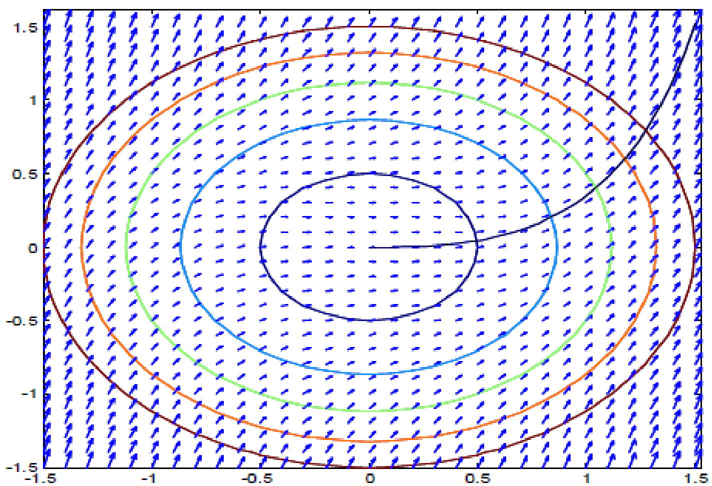
needs to exhibit these two properties: existence and uniqueness of solutions!

Example 1.1 Riccati Equation:

$$\begin{cases} x' = x^2 + t^2 \\ x(0) = 0 \end{cases} .$$

Basic Facts:

1. Not solvable for an explicit form;
   (see reference: G. N. Waston, A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge, 1944, pp. 111-123.)
2. Existence and uniqueness are in fact there
   Geometric way:



Remark 1.1 Riccati Equation shows that we have to change the research direction of ODE and develop some effective methods to explore solution properties and dynamical behaviors on basis of external information $f(t,x)$ without solving ODE.

Example 1.2 The IVP:

$$\begin{cases} x' = \delta(x) \\ x(0) = 0 \end{cases}, \quad \delta(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}.$$

has no solution of this IVP in a traditional sense!

Remark 1.2 Example 1.2 shows that we need at least the continuous property of f for existence – **Peano Theorem. (Schauder Fixed Point Theorem)**

Example 1.3 The IVP

$$\begin{cases} x' = \sqrt{x} \\ x(0) = 0 \end{cases}$$

has two different solutions: $x(t) \equiv 0$ and $x(t) = \frac{t^2}{4}$.

Remark 1.3 Example 1.3 shows that the continuous property of f is not enough for uniqueness of solution. In general, we need an additional condition for sure. Lipschitz condition is a common one. – **Picard Theorem**. (**Banach Fixed Point Theorem**)

Remark 1.3 There are basically **two ways** to prove existence and uniqueness - by a traditional way or using **the fixed point theorems.** For Advanced ODE, one prefers using **the fixed point theorems,** for which we need some of Banach space.

Banach Space

Normed Linear Space

Let X be a linear space, with a norm $\|\cdot\|: X \to R_{\geq 0}$, which satisfies three properties as follows:

(1) $\|x\| \geq 0$; $\|x\| = 0 \Leftrightarrow x = 0$ for any $x \in X$;

(2) $\|\lambda x\| = |\lambda| \|x\|$ for any $x \in X$, $\lambda \in R$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$,

then, X is said a **normed linear space**.

Remark 1.4 $\|\cdot\|: X \to R_{\geq 0}$ is a natural extension of $|\cdot|: R \to R_{\geq 0}$. We can play "convergence" on X with $\|\cdot\|$, not only on R^n in Calculus.

Convergence, Cauchy Sequence

If $\forall\ \varepsilon > 0$, $\exists\ N \geq 1$ s.t. $n \geq N \ \Rightarrow\ \| x_n - x \| < \varepsilon$, whenever $x_n, x \in X$, then, we say that x_n converges to x as $n \to \infty$, denoted by $\lim_{n\to\infty} x_n = x$.

If $\forall\ \varepsilon > 0$, $\exists\ N \geq 1$ s.t. $\| x_n - x_m \| < \varepsilon$ whenever $n, m \geq N$, where $x_n, x_m \in X$, then, we say that $\{x_n\}$ is a **Cauchy sequence**.

Banach Space

A normed linear space X is said to be **complete** if for every $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x \ \Rightarrow\ x \in X$.

A complete normed linear space X is said to be a **Banach space.**

Remark 1.5 Since the norm defined on X is not unique, norm equivalence is significant.

Norm Equivalence

Let $\|\cdot\|$ and $\|\cdot\|_*$ be two norms on X. $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent if there exist $c>0$ and $\overline{c}>0$ s.t.

$$c\|x\|\leq\|x\|_*\leq\overline{c}\|x\|, \text{ for all } x\in X.$$

Proposition 1.1 $C([0,1])$ with $\|f\|_\infty=\sup_{t\in[0,1]}|f(t)|$ is a Banach space. But, $C([0,1])$ with either $\|f\|_1=\int_0^1|f(t)|dt$ or $\|f\|_2=\{\int_0^1 f^2(t)dt\}^{\frac{1}{2}}$ is not Banach.

Proof. First, show that $\| f \|_1$, $\| f \|_2$ and $\| f \|_\infty$ are all norms. (Homework 1)

Second, To show that $X = C([0,1])$ with $\| f \|_\infty$ is Banach, for any Cauchy $\{f_n\}$ in $C([0,1])$ with $\| f \|_\infty$, show $\{f_n(t)\}$ uniformly convergent to $f(t)$ on $[0,1]$ $\Rightarrow$ $f(t) \in C([0,1])$.

Since $\{f_n\}$ is Cauchy, we have

$$| f_n(t) - f_m(t) | \le \sup_{t\in[0,1]} | f_n(t) - f_m(t) | = \| f_n - f_m \|_\infty < \varepsilon \text{, for all } n, m \ge N,$$

$\Rightarrow$ $\{f_n(t)\}$ is Cauchy for any $t \in [0,1]$. So, $\lim_{n\to\infty} f_n(t) = f(t)$ exists on $[0,1]$.

Taking $m \to \infty$, it yields

$$| f_n(t) - f(t) | < \varepsilon \quad \text{for all } n \ge N.$$

Therefore, $\{f_n(t)\}$ is uniformly convergent to $f(t)$ on $[0,1]$.

To show that $X = C([0,1])$ with $\| f \|_1$ and $\| f \|_2$ is not complete.

Define $f_n(t) = \begin{cases} 0, & [0, \frac{1}{2} - \frac{1}{n}] \\ 1, & [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$, $n = 1, 2, \cdots$, linearly interpolating in between. Then,

$$f_n(t) \in C([0,1]) \text{. but } \lim_{n\to\infty} f_n(t) = f(t) = \begin{cases} 0, & [0, \frac{1}{2}) \\ 1, & (\frac{1}{2}, 1] \end{cases} \notin C([0,1]) \quad .$$

By the definitions of $\| f \|_1$ and $\| f \|_2$, for any $m \geq n \geq N$, one has

$$\| f_n - f_m \|_1 \leq \int_0^1 | f_n(t) - f_m(t) | \, dt = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} (\frac{1}{n} - \frac{1}{m}) \leq \frac{1}{n}$$

and

$$\| f_n - f_m \|_2 = \{ \int_0^1 | f_n(t) - f_m(t) |^2 \, dt \}^{\frac{1}{2}} \leq \{ \int_0^1 | f_n(t) - f_m(t) | \, dt \}^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}},$$

$\Rightarrow$ $\{ f_n \}$ is Cauchy both with $\| f \|_1$ and $\| f \|_2$.

Therefore, $C([0,1])$ with $\| f \|_1$ or $\| f \|_2$ is not complete, i.e. not Banach. □

Remark 1.6 R^n with any norm is Banach; $C([0,1])$ is not. Why?

Banach Fixed Point Theorem

Theorem 1.2 Let X be a Banach space with a norm $\|\cdot\|$, $D \subset X$ closed and $\varphi : D \to X$ a map which satisfies

(1) $\varphi(D) \subset D$;

(2) There exists $0 < \alpha < 1$ s.t. $\|\varphi(x) - \varphi(y)\| \le \alpha \|x - y\|$ for all $x, y \in D$.

Then, φ has a unique fixed point x in D, i.e. $\varphi(x) = x$.

Proof. Uniqueness:

If there exist $x, y \in D$ s.t. $\varphi(x) = x$ and $\varphi(y) = y$, then

$$\|x - y\| = \|\varphi(x) - \varphi(y)\| \le \alpha \|x - y\| \text{ with } 0 < \alpha < 1.$$

This is not possible unless $x = y$.

Existence:

Taking $x_0 \in D$, iteration yields the sequence

$$x_1 = \varphi(x_0), \quad x_2 = \varphi(x_1), \quad \cdots, \quad x_{n+1} = \varphi(x_n), \quad \cdots.$$

Note that if there exists $k \geq 1$ s.t. $x_k = \varphi(x_k)$, x_k is desired. If for any $k \geq 1$, $x_k \neq \varphi(x_k)$. Then, $\{x_n\}$ is well defined on D since $x_n \in D$ by $\varphi(D) \subset D$.

To show $\{x_n\}$ is Cauchy on D. Since

$$\| x_{n+1} - x_n \| = \| \varphi(x_n) - \varphi(x_{n-1}) \| \leq \alpha \| x_n - x_{n-1} \|,$$

iterating yields

$$\| x_{n+1} - x_n \| \leq \alpha^n \| x_1 - x_0 \|.$$

If $m > n$, this implies that

$$\| x_m - x_n \| \le \| x_m - x_{m-1} \| + \| x_{m-1} - x_{m-2} \| + \cdots + \| x_{n+1} - x_n \|$$

$$\le (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) \| x_1 - x_0 \|$$

$$\le \frac{\alpha^n}{1-\alpha} \| x_1 - x_0 \| < \varepsilon \quad \text{if} \quad n > N = \left[\ln \frac{\varepsilon(1-\alpha)}{\| x_1 - x_0 \|} \Big/ \ln \alpha \right] + 1 .$$

Therefore, $\{x_n\}$ is Cauchy on D. $\Rightarrow$ $\lim_{n\to\infty} x_n = x \in D$ since D is closed in X, so it is Banach. Then,

$$\varphi(x) = \varphi(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = x .$$

i.e. x is a fixed point of φ. □

Remark 1.7 Solving $f(x)=0 \Leftarrow$ finding a fixed point of $\varphi(x)=x\pm f(x)$!

Remark 1.8 $\varphi(D)\subset D$ ensures $\{x_n\}\in D$ if $x_0\in D$; contractive condition ensures $\lim_{n\to\infty} x_n = x$ and $D\subset X$ closed in X with a norm makes sure $x\in D\subset X$. Any one of them not satisfied may causes the theorem failed.

Schauder Fixed Point Theorem without Proof

Theorem 1.3 Let X be a Banach space, $D \subset X$ is a **closed**, **bounded**, **convex** set and $\varphi : D \to D \subset X$ is **completely continuous**. Then φ has at least one fixed point x in X.

Remark 1.9 D is **convex** means if $x_1, x_2 \in D \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in D,\ 0 \le \lambda \le 1$;

φ is **completely continuous** if φ is **continuous** and for any bounded set $D_1 \subset D$, $\varphi(D_1)$ is **relatively compact** in D; i.e. Any sequence $\{y_n\} \subset \varphi(D_1)$, there exists a **convergent subsequence** in D.

Summary

1) Teaching Program;

2) The norms of finite dimensional Banach space are equivalent like $X = R^n$ with any norm while the norms of infinite dimensional Banach space are not equivalent like $X = C([0,1])$ with $\| f \|_\infty$ and $\| f \|_2$!

3) Fixed point theorems are ready for existence and uniqueness of solutions;

4) Some knowledge of Banach space.

Homework

1) Homework-1.
2) Let $D \subset R^n$ closed and $\varphi : D \to R^n$. Show by examples that

   1. If $\varphi(D) \subset D$ only, φ has not necessarily a fixed point;

   2. If there exists $0 < \alpha < 1$ s.t. $\| \varphi(x) - \varphi(y) \| \leq \alpha \| x - y \|$ for all $x, y \in D$

      only, φ has not necessarily a fixed point.

3) Some properties which are true in finite dimensional Banach spaces like $X = R^n$ are not necessarily true in infinite dimensional Banach spaces like $X = C([0,1])$ with $\| f \|_\infty$! For example:

   1. The closed ball $\{x \in X; \| x \| \le 1\}$ is not necessarily compact!
   2. The Bolzano-Weierstrass theorem that says each bounded sequence has a convergent subsequence is not necessarily true!
   3. "K is compact $\Leftrightarrow$ K is both closed and bounded" is not necessarily true in a Banach space!
      Check with the example as follows:

$$f_n(t) = \begin{cases} 0, & [0,1] / (\frac{1}{(n+1)}, \frac{1}{n}] \\ 1, & \frac{1}{(n+1)} < t \le \frac{1}{n} \end{cases}.$$

4) Review today's lecture contents.

For Math dream, we are together at ECNU!